Dynamics of transverse and longitudinal ionization cooling of muon beams*

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Abstract

Ionization cooling is the best-known cooling mechanism that may reduce the muon beam emittance, before most of the muons decay, to a useful level for the envisioned muon colliders and neutrino factories. We review progress in understanding the linear beam dynamics of both transverse and longitudinal cooling.

1 INTRODUCTION

In order to reduce the transverse and longitudinal emittances of a muon beam for neutrino factories and muon colliders, ionization cooling channels are being developed [1, 2, 3]. For transverse cooling, promising designs consist of continuous straight solenoids to provide strong transverse focusing, liquid hydrogen absorbers at minimum beta locations to provide ionization energy loss, and low-frequency rf accelerating field between the absorbers to provide longitudinal acceleration and focusing. For longitudinal cooling, various emittance exchange schemes have been proposed.

To understand the basic beam dynamics of ionization cooling and to establish a theoretical framework for the design of cooling channels, second-order beam-moment and envelope equations in solenoidal ionization cooling channels are being studied. We adopt this approach because (a) the second moments of beam phase-space distribution contain most of the important beam properties such as rms beam size and angular divergence, and (b) the evolution equations of second moments close on themselves for linear dynamics that often dominate beam evolution.

Beam-moment equations have been developed in several papers for ionization cooling over the past several years [4, 5, 6]. We further developed the beam-moment study emphasizing analytical solutions. There are 10 independent second moments for the two transverse degrees of freedom and 21 for the transverse and longitudinal combined. In general, it is nontrivial to analytically solve such a large number of moment equations. Despite the difficulties, significant progress has been made recently. The first analytical solution was found for the transverse cooling of a cylindrical symmetric beam in a straight solenoidal cooling channel [7]. Then transverse cooling of a general beam was solved [8]. To achieve these, it is important to choose the right parameterization to characterize those moments so that it provides convenient descriptions of beam properties and evolution. Symmetries and associated invariants play a key role in the choice of parameters [9].

Lately our attention has been focused on the dynamics of longitudinal cooling in a bent-solenoid channel. Although the final solution has not been worked out, progress has been made, especially on the dispersion in a bent-solenoid channel that is desired for exchanging the longitudinal emittance to the transverse emittance for cooling. In this short report, we briefly outline the existing theory and progress underway. Readers are encouraged to check out our earlier review [10] as well as more comprehensive and updated presentations.

2 TRANSVERSE COOLING DYNAMICS

The magnetic field in a solenoid is completely determined by the on-axis field strength $B(s)$

$$\mathbf{B}(s,\mathbf{x}) = B(s)\mathbf{e}_s - \frac{1}{2}\frac{dB(s)}{ds}\mathbf{x} + \text{nonlinear terms.} \quad (1)$$

Here $\mathbf{e}_s$ is the unit vector in the s-direction and $\mathbf{x}$ is the transverse displacement. Because of the longitudinal magnetic field, particles with transverse momentum will gyrate around the axis. Thus a particle's motion in the lab frame is fully coupled between the two transverse degrees of freedom and looks rather complicated. Fortunately, beam dynamics is dramatically simplified in the Larmor rotating frame that rotating at 1/2 the cyclotron frequency about the axis. In the Larmor frame, the two transverse motions are decoupled and each degree of freedom can be treated with standard Courant-Snyder theory. Because of the cylindrical symmetry of a solenoidal channel, beam dynamics in the two subspaces are identical. Thus only one set of Twiss parameters ($\hat\beta$,$\hat\alpha$,$\hat\gamma$) is needed to describe the lattice properties, and they satisfy the familiar equations

$$\hat\beta' = -2\,\hat\alpha, \quad \hat\alpha' = \kappa^2\hat\beta - \hat\gamma, \quad \hat\gamma = \frac{1+\hat\alpha^2}{\hat\beta}, \quad (2)$$

with periodic boundary conditions. Here the solenoid lattice is characterized by $\kappa(s) = eB(s)/2p_s$, where p_s is the longitudinal momentum.

Due to the cylindrical symmetry, there are four linearly independent quadratic invariants in a solenoidal channel: the canonical angular momentum $L_z = xP_y - yP_x$ plus three Courant-Snyder-type invariants. Expressed in terms of phase-space coordinates $\{\tilde x,\, \tilde P_x,\, \tilde y,\, \tilde P_y\}$ in the Larmor frame, the Courant-Snyder invarants read

$$I_x = \hat\gamma(s)\tilde x^2 + 2\hat\alpha(s)\tilde x\tilde P_x + \hat\beta(s)\tilde P_x^2, \quad (3)$$

$$I_y = \hat\gamma(s)\tilde y^2 + 2\hat\alpha(s)\tilde y\tilde P_y + \hat\beta(s)\tilde P_y^2, \quad (4)$$

$$I_{xy} = \hat\gamma(s)\,\tilde x\,\tilde y + 2\hat\alpha(s)\frac{\tilde x\tilde P_y + \tilde y\tilde P_x}{2} + \hat\beta(s)\tilde P_x\tilde P_y. \quad (5)$$

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The last invariant is unique to the cylindrically symmetric solenoidal channels.

Using these four linearly independent quadratic invariants, the equilibrium Gaussian distribution can be written as

$$\rho(\tilde{X}) = \frac{1}{(2\pi)^2\sqrt{\epsilon_{4D}}} e^{-\frac{\epsilon_y I_x + \epsilon_x I_y - 2\epsilon_{xy} I_{xy} - L L_z}{2(\epsilon_x \epsilon_y - \epsilon_{xy}^2 - L^2/4)}}, \quad (6)$$

and the 4D emittance is

$$\epsilon_{4D} \equiv \det \Sigma = (\epsilon_x \epsilon_y - \epsilon_{xy}^2 - L^2/4)^2. \quad (7)$$

Here the four parameters L, ϵ_x, ϵ_y, ϵ_{xy} are the beam angular momentum and emittances that can be written as

$$L \equiv \langle L_z \rangle = \langle \tilde{x}\tilde{P}_y \rangle - \langle \tilde{y}\tilde{P}_x \rangle, \quad (8)$$

$$\epsilon_x \equiv \sqrt{\langle \tilde{x}^2 \rangle \langle \tilde{P}_x^2 \rangle - \langle \tilde{x}\tilde{P}_x \rangle^2} = \frac{1}{2}\langle I_x \rangle, \quad (9)$$

$$\epsilon_y \equiv \sqrt{\langle \tilde{y}^2 \rangle \langle \tilde{P}_y^2 \rangle - \langle \tilde{y}\tilde{P}_y \rangle^2} = \frac{1}{2}\langle I_y \rangle, \quad (10)$$

$$\epsilon_{xy} \equiv \sqrt{\langle \tilde{x}\tilde{y} \rangle \langle \tilde{P}_x\tilde{P}_y \rangle - \langle \frac{\tilde{x}\tilde{P}_y + \tilde{y}\tilde{P}_x}{2} \rangle^2} = \frac{1}{2}\langle I_{xy} \rangle. \quad (11)$$

For an empty solenoidal channel, these four beam parameters are conserved quantities. To respect the cylindrical symmetry, it is often convenient to use $\epsilon_s = (\epsilon_x + \epsilon_y)/2$ and $\epsilon_a = (\epsilon_x - \epsilon_y)/2$, the rotationally symmetric and asymmetric parts of the transverse emittances.

When absorbers and rf fields are applied, ionization cooling takes place. Assuming the reference particle is kept at a constant momentum p_s, based on moment equations, the four beam parameters evolve as [7, 8]

$$\frac{d}{ds}\begin{bmatrix} \epsilon_s \\ L \end{bmatrix} = -\eta \begin{bmatrix} 1 & -\kappa\hat{\beta} \\ -\kappa\hat{\beta} & 1 \end{bmatrix} \begin{bmatrix} \epsilon_s \\ L \end{bmatrix} + \begin{bmatrix} \hat{\beta}\chi \\ 0 \end{bmatrix}, \quad (12)$$

$$\frac{d\epsilon_a}{ds} = -\eta\, \epsilon_a \,, \quad \frac{d\epsilon_{xy}}{ds} = -\eta\, \epsilon_{xy}. \quad (13)$$

Here, the ionization dumping is characterized by

$$\eta(s) = \frac{1}{p_s v}\frac{dE}{ds}\bigg|_{loss} = \frac{1}{p_s}\frac{dp}{ds}\bigg|_{loss}, \quad (14)$$

where $dE/ds|_{loss}$ is a positive quantity representing the energy loss per unit length in the absorber material, and v is the muon's speed. The multiple-scattering heating is characterized by

$$\chi(s) = \left(\frac{13.6\,\mathrm{MeV}}{p_s v}\right)^2 \frac{1}{L_{Rad}}, \quad (15)$$

where L_{Rad} is the radiation length of the material. The beam evolution equations have been solved analytically. Thus the linear dynamics of transverse cooling is solved analytically.

The transverse cooling behavior in a periodic cooling channel can be summarized as: 1) incoming beam quickly matches into a periodic cooling channel via filamentation and particle loss; 2) the asymmetric part of the matched beam will be cooled away exponentially without heating; 3) the symmetric emittance and angular momentum will be cooled by ionization energy loss while heated by multiple scattering; and 4) the final equilibrium state is a round beam with only symmetric emittance determined by the balance of cooling and heating (may have net angular momentum depending on channel design). The effects of longitudinal dynamics on the transverse cooling have not been addressed yet. Figure 1 compares the analytical calculation and tracking simulations (including all effects) for a periodic solenoidal cooling channel.

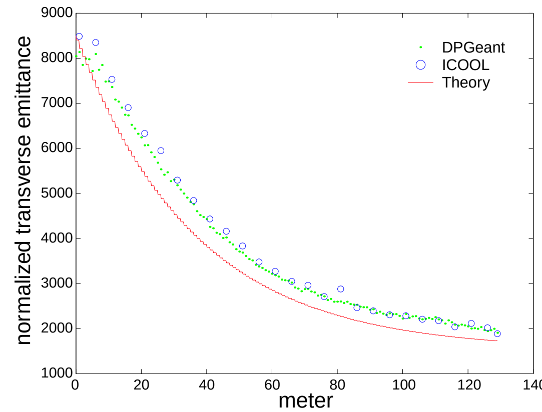

Figure 1: Transverse cooling calculated analytically, with ICOOL, and with DPGeant. (Simulation courtesy of P. Lebrun; reproduced with permission from APS, PRL, **84**.)

3 FORMULAS FOR BETA FUNCTION AND ORBIT STABILITY

A major task in transverse cooling channel design is to generate a focusing lattice that has small beta function and large momentum acceptance [11] using achievable solenoid fields. Since continuous focusing is necessary to contain the divergent muon beam, the solenoid magnets cannot be treated as a few simple lenses. To facilitate lattice design, analytical formulas for beta function and orbit stability are derived in ref. [12]. The lattice properties are determined by the Hills equation $x'' + K(s)x = 0$ with $K(s) = \kappa(s)^2$. For a periodic solenoidal channel, the on-axis field $B(s)$ varies continuously with period L. It is natural to use the Fourier coefficients $\{\vartheta_n\}$ of the normalized focusing strength function $\vartheta(\varsigma) = \left(\frac{L}{\pi}\right)^2 K(\frac{L}{\pi}\varsigma)$ to characterize the solenoid field. Here $\varsigma = \pi\frac{s}{L}$ is the normalized position. The beta function can be calculated with

$$\beta(s) = \frac{L}{\pi}\frac{\sin(\sqrt{\vartheta_0}\pi)}{\sqrt{\vartheta_0}\sin\mu}\left[1 + \sum_{n=1}^{\infty}\frac{\Re[\vartheta_n e^{i2n\pi s/L}]}{n^2 - \vartheta_0} + \cdots\right]. \quad (16)$$

Here μ is the one-period phase advance that can be calculated via $\cos\mu = \Delta/2$. And Δ is the trace of the one-period transfer matrix that can be calculated with

$$\Delta = 2\cos(\sqrt{\vartheta_0}\pi) + \frac{\pi\sin\sqrt{\vartheta_0}\pi}{2\sqrt{\vartheta_0}}\sum_{n=1}^{\infty}\frac{|\vartheta_n|^2}{\vartheta_0 - n^2} + \cdots. \quad (17)$$

The orbit stability can be determined by the well-known criteria $|\Delta| < 2$. Higher-order expressions of β and Δ are available in ref. [12]. Using these formulas one can quickly estimate the basic properties of a solenoidal channel from the Fourier coefficients of its focusing function. Furthermore, insight can be gained from these analytical expressions. For example, the form of Eq. (16) already suggests a few general strategies for obtaining a small $\beta(s)$: 1) shorter period; 2) stronger field (resulting in larger ϑ_0 and higher-order passband); 3) larger phase-advance term $\sin\mu$; and 4) larger harmonics to cancel the unity term within the brackets of Eq. (16).

4 LONGITUDINAL COOLING DYNAMICS

Longitudinal cooling is much harder to achieve. The ionization process itself does not effectively cool the beam momentum spread because the energy-loss rate is not sensitive to beam momentum except for very low-energy muons. Despite numerous proposals and simulation studies, there are no solid demonstration of a working scheme. A promising option is the “emittance exchange” scheme: introducing dispersion to spatially separate muons of different energies and then using wedge absorbers to discriminatively cool them. Our present focus is to upgrade an effective transverse cooling channel such that both longitudinal and transverse degrees of freedom can be cooled via emittance exchange.

To generate dispersion in a solenoid channel, the straightforward approach is to superimpose dipole fields with the solenoid field and make the solenoids bend along the curved reference orbit determined by the dipole field. Neither the solenoid nor the dipole field can be treated as piecewise constant blocks, and the lattice consists of rather complicated combined function magnets. Since the main solenoid field continuously rotates the beam and tends to make the beam rotationally symmetric, it may be advantageous to have symmetric focusing. To achieve this, gradient dipoles (with field index n=1/2) could be used.[1]

In terms of the Hamiltonian, the bent-solenoid model under investigation can be written as

$$\begin{aligned} H \;=\; & \frac{1}{2}\left(p_x^2+p_y^2\right)+\frac{1}{2}\kappa(s)^2\left(x^2+y^2\right)+\kappa(s)L_z \qquad (18)\\ & -\frac{x\delta}{\rho(s)}+\frac{x^2}{2\rho(s)^2}-\frac{1}{2}k(s)\left(x^2-y^2\right)+\frac{1}{2}\left[\delta^2+\mathrm{v}(s)z^2\right], \end{aligned}$$

where $\{z,\delta\}$ are the canonical variables for the longitudinal phase space, $\rho(s)$ is the bending radius, $k(s)$ is the quadrupole strength, and $\mathrm{v}(s)$ is the longitudinal focusing strength. The first line corresponds to the straight solenoidal channel that has been solved for the transverse cooling. For a gradient dipole with symmetric focusing, $1/\rho(s)^2 - k(s) = k(s)$, i.e., the quadrupole component is tied to the bending radius as $k(s) = 1/2\rho(s)^2$. The total focusing strength becomes $K(s) = \kappa(s)^2 + 1/2\rho(s)^2$. The Hamiltonian reduces to

$$\begin{aligned} H \;=\; & \frac{1}{2}\left(p_x^2+p_y^2\right)+\frac{1}{2}K(s)\left(x^2+y^2\right)+\kappa(s)L_z \qquad (19)\\ & -\frac{x\delta}{\rho(s)}+\frac{1}{2}\left[\delta^2+\mathrm{v}(s)z^2\right]. \end{aligned}$$

In the Larmor frame, the dispersions ($\tilde{D}_x$, $\tilde{D}_y$) of this system have been worked out to satisfy

$$\tilde{D}_x'' + K(s)\tilde{D}_x = \frac{\cos[\theta(s)]}{\rho(s)}, \qquad (20)$$

$$\tilde{D}_y'' + K(s)\tilde{D}_y = \frac{\sin[\theta(s)]}{\rho(s)}, \qquad (21)$$

where $\theta(s) = \int_0^s \kappa(\bar{s})d\bar{s}$ is the rotating angle of the Larmor frame. These equations lay the foundation for the dispersion design of a symmetrically focused bent-solenoid channel. Note that the r.h.s. driving term is not independent of the focusing strength $K(s)$. This makes it difficult to find a dispersion solution. Nonetheless, example solutions have been demonstrated.

Using dispersion functions, the transverse and longitudinal motions can be decoupled. This paves the way to analytically solve the beam dynamics. Moment equations for the bent-solenoid channel have also been developed. Effort is being made to reduce the 21 moment equations to a much smaller number of envelope equations. Hopefully, analytical solutions for the longitudinal cooling can be obtained in the near future. There is still much to learn about longitudinal cooling.

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